Queen Mary, University of London
LOGIC I: Mathematical writing
Coursework 2

to be handed in on Monday 26 January 2004.

1. Your local newspaper runs a column answering readers' questions. Readers have written in with questions, and the newspaper has invited you to choose one of the questions and write a reply for their readers, in not more than 200 words. (Fewer would be acceptable.) Choose one of the following questions to answer:

   A. When we look at ourselves in a mirror, why do we see ourselves with right and left reversed, but not upside down?

   B. Somebody told me a joke that when Russian mathematicians left Russian universities and got jobs in British universities, the average quality of the researchers went up in both British and Russian universities. I don't understand this joke; what does it mean?

2. Each of the following is a theorem in a published mathematical text, and in each case the theorem is followed by its proof. From the forms of the theorems, guess the opening words of the proof, and state the RTP. If you can say anything about the likely overall structure of the proof, say that too. (You are not expected to understand the mathematical details!)

   (a) Theorem. Every transitive permutation representation of G is equivalent to one on the right cosets of a subgroup of G.
   (b) Theorem. A hyperboloid of one sheet contains two families of generating lines.
   (c) Theorem. If G possesses a faithful irreducible representation on a vector space over a field of characteristic p, then G has no nontrivial normal p-subgroups.
   (d) Theorem. If an S_p-subgroup of G lies in the center of its normalizer in G, then G has a normal p-complement.
   (e) A parallel projection maps straight lines to straight lines. [It may help to know that a parallel projection is a kind of function.]

(f) Theorem. Let V be a finite-dimensional linear space. Then every finite basis for V has the same number of elements.

(g) THEOREM. There are no positive integral solutions of $x^4 + y^4 = z^2$.

(h) Let Y be a random variable with finite mean μ and variance σ^2. Then for any positive constant k, $P(|Y - \mu| < k\sigma) \geqslant 1 - 1/k^2$.

(i) Affine transformations map ellipses to ellipses, parabolas to parabolas and hyperbolas to hyperbolas.

(j) A necessary and sufficient condition that a number $m > 1$ should be prime is that $m|(m-1)! + 1$.

3. Rewrite each of the following to make clear that it is an existence statement.

(a) n is even.

(b) n is not prime.

(c) The 2×2 matrix M over $\mathbb{R}$ is invertible.

(d) The sets X and Y are not disjoint.

(e) The triangle ABC is isosceles.

Wilfrid Hodges, 18 January 2004

Queen Mary, University of London
LOGIC I: Mathematical writing
Coursework 7

to be handed in on Monday 1 March 2004.

1. You are the leader of a team of mathematics teachers. Here are two definitions of what it means for a sequence (a_n) of real numbers to converge to a limit ℓ.

    (a) (a_n) converges to ℓ if and only if for every positive real ε there is a positive integer N such that $|a_n - \ell| < \varepsilon$ whenever $n \geqslant N$.

    (b) We say that (a_n) converges to ℓ if when n increases, a_n gets as close as you like to ℓ.

    Definition (a) is precise but people often find it hard to understand. You want to point out to your team that (b), although it seems easier to understand, could lead students to the wrong answer about whether a particular sequence (a_n) converges to a particular number ℓ. Write a note for your team about this, using not more than 300 words; it should include an example and a discussion of how (b) could mislead people about the example.

2. We write $\mathbb{N}$ for the set of natural numbers $0, 1, 2, \ldots$. A function $f(x, y)$ is called a *one-one correspondence from* $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$ if:

    (a) $f(k, m)$ is defined if and only if k and m are natural numbers;

    (b) if k, m are natural numbers then $f(k, m)$ is a natural number;

    (c) if n is a natural number then there is exactly one pair (k, m) of natural numbers such that $f(k, m) = n$.

    Write a textbook proof that there is a one-one correspondence from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$, *without using any diagrams*. Try to make use of the following hint from Paul Halmos, 'Naive Set Theory' p. 92:

    "One way to prove this is to write down the elements of $\mathbb{N}$

in an infinite array by counting down the diagonals, thus:

0	1	3	6	10	15	...
2	4	7	11	16	...	
5	8	12	17	...		
9	13	18	...			
14	19	...				
20	...					
...						

... the details are easy to fill in."

(So you have to extract from the array a mathematical idea that can be used without actually drawing the array. There are several possible ways of doing this. You can use a recursive definition, though in fact there is a fairly simple closed form definition. To get started, how is $f(i, j)$ related to $f(i-1, j+1)$ when $i > 0$?)

Wilfrid Hodges 22 February 2004

Queen Mary, University of London
LOGIC I: Mathematical writing
Coursework 9

to be handed in on Monday 22 March 2004.
(NB: A week is left free for preparing your oral presentation).

(a) You will be given a copy of the article

> Liz Bills and David Tall, 'Operable definitions in advanced mathematics: The case of the least upper bound', *Proceedings of PME 22, Stellenbosch, South Africa,* 2 (1998) 104–111.

After reading this article, write an essay (not more than 1000 words) on how you would explain the concept of least upper bound in a textbook for first year students. You can assume that the students have A level calculus and that they understand the notion of a linear ordering. If you like, you can write a section of your textbook and then make some comments on it and how it reflects points made by Bills and Tall.

(b) Say what is wrong with the following two arguments.

(i) THEOREM. There is a real number x such that for every y, $x + y = x$.
PROOF. We know $x + 0 = x$. Let $y = 0$; then we have immediately

$$x + y = x.$$

(ii) $(0, 0)$ is a point of order 2 on the given curve, since any point of order 2 has the form $(x, 0)$, where x is a root of the cubic equation $0 = x^3 + ax$.

(Suppes gives (i) as an example of a fallacious proof, in *Introduction to Logic,* p. 142. (ii) is essentially a quote from a textbook of elliptic curves; you can detect the mistake without knowing the context.)

Wilfrid Hodges, 7 March 2004