An experimental course in mathematical writing

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This year I taught a new second-year undergraduate course whose main aim was to help the students to write and read mathematics intelligently. I made two design decisions at the outset. First, the course should be aimed at middle-level students; it shouldn't rely on the students being high fliers. Second, there should be some substantial mathematics in the course, otherwise it could hardly teach *mathematical* writing.

What mathematics to put in?

One can easily find suitable 'puzzle' questions, starting with the old brainteaser:

> *When we look at ourselves in a mirror, why do we see ourselves reversed left to right but not upside down?*

When the students have made their first attempt at this, one can show them Martin Gardner's account ([2] p.23f). Not all students appreciate that their answer needs to point to some feature that the left-right dimension *does* have and the up-down dimension *doesn't*. So there is an issue of logic that we need to discuss, and at this point several students ask to have another shot at the question.

But for me, and I suspect for most students too, a random selection of good essay topics doesn't provide enough structure for a course. Sally Mitchell, who leads the college's initiative *Writing in the Disciplines*, gave me a valuable clue here. She showed me her own work on using informal logic to give the students a framework for writing [4]. The informal logic that she was using seemed to me too far removed from the arguments that a mathematics student needs to handle. For example it said nothing about making and using assumptions, two essential tools of mathematical thinking. So I thought of taking her approach but with a more sophisticated logic. Hence the name of the course: 'Logic I: Mathematical Writing'.

The backbone of the course is a study of informal reasoning in mathematics. We start with a short review of Blaise Pascal's *De l'Esprit Géométrique* [5], and the two main topics of the course (both introduced by Pascal) are defining and proving. Each of these topics gives scope for both practical examples and background theory. For example one question in the exam asked the students to describe in words the function with the graph shown in Fig 1 below:

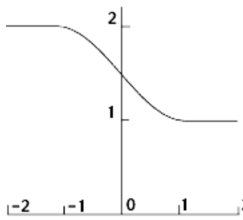

Fig 1 Students asked to describe function in words

and then to give a mathematical definition of such a function. Most students don't instinctively see the need for a definition by cases, and those who do see it don't always appreciate that they must ensure their function has exactly one value at each point of the interval.

Examples like this lead naturally into a general discussion of what you need to specify when you describe a mathematical object. For instance, how do you show that the pattern in Fig 2 has an orientation?

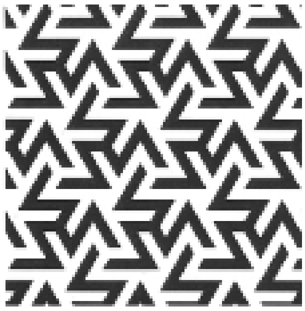

Fig 2 Show that this pattern has an orientation

Here the students have to discover some feature whose mirror image doesn't also appear in the pattern, and then they must describe this feature in words. We check whether the other students can understand their definition.

At the more theoretical end, the students learn to recognise and manipulate various types of definition, including some straightforward types of definition by induction. Our third year logic course, Logic II, introduces a number of definitions by induction, for example to define the set of formulas of a logical language. It will be interesting to see if the treatment in Logic I helps the students who go on to Logic II.

Under the head of proving, we ask what kinds of proof are needed to prove various kinds of statement. For example, how would we expect a proof of the following theorem to start?

If V is a finite-dimensional linear space, then every finite basis for V has the same number of elements.

The proof needs to draw out the assumption:

Let V be a finite-dimensional linear space.

Then we expect:

Let A and B be finite bases of V.

Why do we introduce names for bases of V? Why two of them?

To answer these questions, the students didn't need to know what a linear space is. (I'm not sure any of them did know; in London we call these objects 'vector spaces'.) This observation leads us on to study some of the main shapes of mathematical arguments, noting that we can recognise the shapes even if we have no idea what the terms mean. Feynman's beautiful remarks ([1] p.85) about how to bluff your way through a mathematical discussion come in handy here:

As [the mathematicians] are telling me the conditions of the theorem, I construct something that fits all the conditions. You know, you have a set (one ball)-disjoint (two balls). Then the balls turn colours, grow hairs... Finally they state the theorem, which is some dumb thing about the ball which isn't true for my hairy green ball thing, so I say "False!"

Feynman's remarks point at once to axiomatic arguments where the terms are quite literally meaningless. In the exam the students have to construct proofs or counter examples for some axiomatic entailments written in English.

Only one mathematician ever won a Nobel Prize for Literature: Bertrand Russell. One of our more successful exercises was to rewrite a passage from his *Introduction to Mathematical Philosophy* [7] in a modern idiom. The content of the passage was relevant to the logic in the course.

Another useful rewriting exercise was to bring an argument of Diophantus up to date. Diophantus states much less than he proves, so the students have to analyse the proof and see exactly what it does establish. Diophantus doesn't always remember to say what kinds of numbers he is talking about, so the students have to consider the range of numbers for which the proof holds.

We also rewrote some plain bad arguments. Standard examples are:

1. The proof deduces A from B when it should have deduced B from A. (The students found this mistake surprisingly difficult to detect and repair.)
2. The proof lapses into unexplained symbolism.
3. The proof uses unstated assumptions.

There are a range of examples of this third error. Maybe it isn't an error when the assumptions are well-known properties of the plane; we reviewed some examples of 'visual' proofs to uncover their assumptions.

The last week of the course discussed the rights and wrongs of using other people's material. This led to an exam question:

Your company has hired two recent graduates, and they have been asked to write reports that may involve searching the world-wide web for information. Write a memo to warn them of the main dos and don'ts when using or quoting material from the web. Use not more than 400 words.

This kind of question revealed that many mathematical students have very little idea of the social side of writing. I encouraged the students to think how to start the memo with suitable words of welcome or encouragement.

The course aimed to put various mathematical words into the students' active vocabularies. For this we had weekly discussion sessions, where the students had to explain their ideas to the other students. The students also had to give class presentations, which my colleague Franco Vivaldi and I assessed together. Though we had no trouble on agreeing our assessments (and I seize this opportunity to thank Franco for his help and advice - he has been teaching his students to write mathematical English for years now), I don't think my instructions to the students were well enough prepared. I shall have to think some more about the purpose of the presentations.

Coursework and Assessment

Coursework was set weekly. Through the first half of the course, assessment of the coursework was difficult and time-consuming. Coursework and the class presentation together counted for half the assessment, so this mattered. This coming year I shall decide in advance exactly how many marks will be awarded for this or that aspect of the work. Explaining the mark scheme to the students will probably help in teaching the material and it should reassure students who are bewildered about the purpose of the course.

Of course the assessment must measure whether the students' writings are grammatical, well structured, clear and relevant. We discussed structure and clarity constantly through the course, and the students often disagreed with me or each other about whether something was clear. One thing I didn't find time to do this year was to put together a short list of pointers to good writing style. But we did spend a day paraphrasing parts of a very pertinent paper by my colleague Donald Preece [6], about right and wrong ways of expressing statistical facts.

It would be good to have a textbook for the course. I found nothing at undergraduate level that was specifically about mathematical writing. For mathematical reasoning I found two books with helpful material, Steven Lay [3] and Patrick Suppes [8], but neither of them exactly fits the bill. Maybe if the fates allow, I will write something. In the meanwhile the course material, the coursework assignments and the syllabus are on the web at:

www.maths.qmul.ac.uk/~wilfrid/logicone

References

[1] Richard P Feynman, *Surely You're Joking Mr Feynman!*, Vintage 1985.
[2] Martin Gardner, *The Ambidextrous Universe*, Penguin 1967.
[3] Steven R Lay, *Analysis with an Introduction to Proof*, Prentice Hall 2001.
[4] Sally Mitchell, 'Putting argument into the mainstream', in *Learning to Argue in Higher Education*, ed. Sally Mitchell and Richard Andrews, Boynton/Cook 2000.
[5] Blaise Pascal, *De l'Esprit Géométrique*, in *Oeuvres Complètes*, Seuil 1963, pp. 348-355.
[6] Donald Preece, 'The language of size, quantity and comparison', *The Statistician* 36 1987 pp. 45-54.
[7] Bertrand Russell, *Introduction to Mathematical Philosophy*, George Allen and Unwin 1919.
[8] Patrick Suppes, *Introduction to Logic*, Dover 1999.